1] $y'' + 4y' + 3y = 0 \qquad 0 < x < 1$

$y(0) = 1, \; y(1) = 2$

$r^2 + 4r + 3 = 0$

$(r+1)(r+3) = 0$

$y = c_1 e^{-x} + c_2 e^{-3x}$

$y(0) = c_1 + c_2 = 1 \quad \Rightarrow \quad c_2 = 1 - c_1$

$y(1) = \frac{c_1}{e} + \frac{c_2}{e^3} = 2$

$$\frac{c_1}{e} + \frac{(1-c_1)}{e^3} = 2$$

$$\left(\frac{1}{e} - \frac{1}{e^3}\right) c_1 = 2 - \frac{1}{e^3}$$

$$c_1 = \frac{2 - \frac{1}{e^3}}{\frac{1}{e} - \frac{1}{e^3}} \cdot \frac{e^3}{e^3}$$

$$c_1 = \frac{2e^3 - 1}{e^2 - 1}$$

$$c_2 = 1 - \frac{2e^3 - 1}{e^2 - 1} = \frac{e^2 - 1 - 2e^3 + 1}{e^2 - 1}$$

$$c_2 = \frac{e^2 - 2e^3}{e^2 - 1}$$

$$\boxed{y = \frac{2e^3 - 1}{e^2 - 1} e^{-x} + \left(\frac{e^2 - 2e^3}{e^2 - 1}\right) e^{-3x}}$$

2] $y'' + y = x^2$

$y(0) = 1, \; y(1) = 2$

$y_c'' + y_c = 0 \implies y_c = C_1 \cos x + C_2 \sin x$

$Y = Ax^2 + Bx + c$ Particular solution

$Y' = 2Ax + B$

$Y'' = 2A$

$Y'' + Y = Ax^2 + Bx + C + 2A = x^2 + 0 \cdot x + 0 \cdot 1$

$\implies A = 1$

$B = 0$

$C + 2A = 0 \qquad C = -2A = -2$

$y = C_1 \cos x + C_2 \sin x + x^2 - 2$

$y(0) = C_1 \cdot 1 + C_2 \cdot 0 + 0^2 - 2 = 1 \implies C_1 = 3$

$y(1) = C_1 \cos 1 + C_2 \cdot \sin 1 + 1^2 - 2 = 2$

$C_1 \cos 1 + C_2 \sin 1 = 2 + 2 - 1 = 3$

$3 \cos 1 + C_2 \sin 1 = 3$

$C_2 = \dfrac{3 - 3\cos 1}{\sin 1}$

$= 3(\csc 1 - \cot 1)$

$y = 3 \cos x + 3(\csc 1 - \cot 1) \sin x + x^2 - 2$

3] Now we are asked to solve

$$y'' + y = x^2 \qquad 0 < x < L$$

$$y(0) = 1, \; y(L) = 2$$

The problem is the same up to the point where we find $y(L)$

$$y = 3\cos L + c_2 \sin L + L^2 - 2 = 2$$

$$\sin L \cdot c_2 = 4 - L^2 - 3\cos L$$

$$c_2 = \frac{4 - L^2 - 3\cos L}{\sin L}$$

has a unique solution as long as $\sin L \neq 0$

i.e. as long as $L \neq n\pi$

4] $$2x^2 y'' + 3xy' - y = x \qquad 1 < x < 4$$

$$y(1) = 5, \quad y(4) = 4$$

Given that $y_1 = x^{-1}$ & $y_2 = x^{1/2}$ form a fundamental solution set of the homogeneous problem

$$W = y_1 y_2' - y_1' y_2$$

$$= x^{-1} \cdot \tfrac{1}{2} x^{-1/2} - (-x^{-2}) x^{1/2}$$

$$= \tfrac{1}{2} x^{-3/2} + x^{-3/2}$$

$$= \tfrac{3}{2} x^{-3/2}$$

IMPORTANT POINT: THE VARIATION OF PARAMETERS METHOD IS POSED in the form

$$y'' + p(x) y' + q(x) y = f(x)$$

so before we can begin, we need to divide the equation by $2x^2$, which yields

$$y'' + \frac{3x}{2x^2}y' - \frac{1}{2x^2}y = \frac{x}{2x^2} = \frac{1}{2x} \quad \text{so } f(x) = \frac{1}{2x}$$

Then the particular solution is

$$Y = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where $u_1' = -\frac{y_2(x)f(x)}{W(x)} = \frac{-\sqrt{x}\cdot\frac{1}{2x}}{\frac{3}{2}x^{-3/2}} = \frac{-\frac{1}{2\sqrt{x}}}{\frac{3}{2x^{3/2}}} \cdot \frac{2x^{3/2}}{2x^{3/2}}$

$$u_1' = -\frac{x}{3}$$

$$\boxed{u_1 = -\frac{x^2}{6}}$$

$$u_2' = \frac{y_1 f(x)}{W(x)} = \frac{\frac{1}{x}\cdot\frac{1}{2x}}{\frac{3}{2x^{3/2}}} \cdot \frac{2x^{3/2}}{2x^{3/2}} = \frac{1}{3}x^{-1/2}$$

$$u_2 = \frac{1}{3}\cdot 2x^{1/2}$$

$$u_2 = \frac{2}{3}\sqrt{x}$$

$$Y = u_1y_1 + u_2y_2 = \frac{-x^2}{6}\cdot\frac{1}{x} + \frac{2}{3}\sqrt{x}\cdot\sqrt{x}$$

$$= -\frac{1}{6}x + \frac{2}{3}x$$

$$= \frac{x}{2}$$

Then the full solution is

$$y = c_1 \cdot \frac{1}{x} + c_2\sqrt{x} + \frac{x}{2}$$

The boundary conditions are thus

$$y(1) = C_1 \cdot 1 + C_2 \cdot 1 + \frac{1}{2} = 5$$

$$y(4) = C_1 \cdot \frac{1}{4} + C_2 \cdot \sqrt{4} + \frac{1}{2} \cdot 4 = 4$$

$$C_1 + C_2 = \frac{9}{2} \qquad C_2 = \frac{9}{2} - C_1$$

$$\frac{1}{4}C_1 + 2C_2 = 2$$

$$\frac{1}{4}C_1 + 2\left(\frac{9}{2} - C_1\right) = 2$$

$$\frac{1}{4}C_1 + 9 - 2C_1 = 2$$

$$-\frac{7}{4}C_1 = -7$$

$$C_1 = 4$$

$$C_2 = \frac{9}{2} - 4 = \frac{1}{2}$$

$$\boxed{y = \frac{4}{x} + \frac{1}{2}\sqrt{x} + \frac{x}{2}}$$