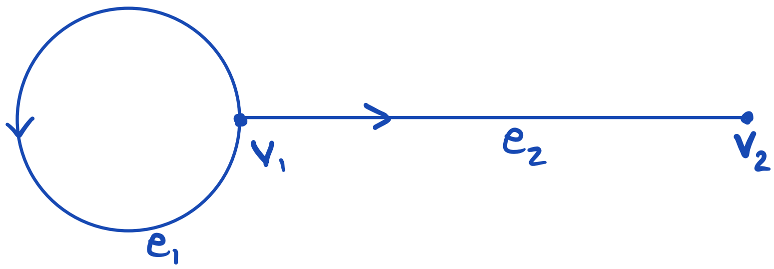
LECTURE 1: THE POISSON PROBLEM

let G be a compact graph

$$G = (\mathbb{V}_G, \mathbb{E}_G), \quad \|\mathbb{V}_G\| = \#\text{vertices}, \quad \|\mathbb{E}_G\| = \#\text{edges}$$



LET $\mathbb{E}_G^k = \{$EDGES INCIDENT ON VERTEX v_k, COUNTED WITH MULTIPLICITY & INCOMING/OUTGOING$\}$

$= \{e_{i_1}, \dots, e_{i_{d_k}}\}$

WHERE d_k = DEGREE OF VERTEX k

POISSON PROBLEM

$$\Delta u = f$$

SUBJECT TO VERTEX CONDITIONS AT VERTEX v_k

$$\left.\begin{aligned} & u_{i_1} = u_{i_2} = \dots = u_{i_{d_k}} \equiv U_k \\ & \sum_{e_j \in \mathbb{E}_G^k} \frac{d}{dx} u_{i_j}(0) + \alpha_k U_k = \varphi_k \end{aligned}\right\} \quad d_k \text{ EQUATIONS}$$

EACH EDGE HAS TWO ENDPOINTS, SO THERE ARE

$$2 \cdot \|\mathbb{E}_G\| = \sum_{k=1}^{\|\mathbb{V}_G\|} d_k \quad \text{LINEAR EQUATIONS TO SATISFY}$$

THE SECOND VERTEX CONDITION IS ANALOGOUS TO A ROBIN CONDITION. IF $\alpha_k = \varphi_k = 0$ THIS IS THE NEUMANN-KIRCHHOFF VERTEX CONDITION. CAN BE THOUGHT OF AS A ZERO NET-FLUX CONDITION.

A SIMPLER PROBLEM

$$u''(x) = f(x) \quad , \quad u(0) = u(1) = 0$$
$$0 < x < 1 \qquad\qquad f \in C^2([0,1])$$

APPROXIMATION VIA FINITE DIFFERENCES

$$u(x+h) = u(x) + h u'(x) + \frac{h^2}{2} u'' + \frac{h^3}{6} u''' + \frac{h^4}{4!} u^{(4)}(\xi_+) \quad , \quad x < \xi_+ < x+h$$
$$u(x-h) = u(x) - h u'(x) + \frac{h^2}{2} u'' - \frac{h^3}{6} u''' + \frac{h^4}{4!} u^{(4)}(\xi_-) \quad , \quad x < \xi_+ < x+h$$

$$u(x+h) - 2u(x) + u(x-h) = h^2 u'' + \frac{h^4}{4!}\left(u^{(4)}(\xi_+) + u^{(4)}(\xi_-)\right), \quad x-h < \xi_- < x$$

$$u'' = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} - \frac{h^2}{12} u^{(4)}(\xi), \quad x-h < \xi < x+h$$

FIX $h = \frac{1}{N}$, let $x_n = nh$, $n = 0, \ldots, N$

LET $u_n \approx u(x_n)$, $f_n = f(x_n)$, $n = 1, \ldots, N$

$u_0 = u_N = 0$ SPECIFY BOUNDARY CONDITIONS EXPLICITLY

$$\frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} = f_n \qquad n = 1, \ldots, N$$

DEFINE $D_2 = \frac{1}{h^2}\underbrace{\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & & \ddots & & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}}_{N-1}\Bigg\}{N-1}$, $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$, $\vec{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_{N-1} \end{bmatrix}$

THEN $D_2 \vec{u} = \vec{f}$

D_2 IS TRIDIAGONAL & SYMMETRIC

IS ACCURATE TO $O(h^2)$, i.e. if h^2 SUFF SMALL

THEN $|u_n - u(x_n)| \leq C \cdot h^2 \max_{0 \leq x \leq 1} |u^{(4)}(x)|$

SUPPOSE WE REPLACE THE BOUNDARY CONDITIONS

$$u'(0) = 0$$

$$u'(1) = 1$$

NOW $u(0)$ & $u(1)$ ARE UNKNOWN

HOW TO APPROXIMATE $u'(0)$

$$u'(0) = \frac{u(h) - u(0)}{h} + O(h)$$

CAN USE THE APPROXIMATION

$$u'(0) = \frac{u_1 - u_0}{h} = 0 \Rightarrow u_0 = u_1$$

SIMILARLY $u_N = u_{N+1}$

THIS MODIFIES $D_2 \to D_2' = \frac{1}{h^2}\begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & & \ddots & & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{bmatrix}$

NOTE D_2' IS STILL TRIDIAGONAL + SYMMETRIC

BUT IF $\vec{u} = D_2' \vec{f}$

then $(u_n - u(x_n)) = O(h)$

FIRST-ORDER DUE TO $O(h)$ APPROXIMATION IN BOUNDARY CONDITIONS

TO MAINTAIN 2ND-ORDER ACCURACY

$$\left.\frac{du}{dx}\right|_{x_0} = \frac{-3u_0 + 4u_1 - u_2}{2h} + \frac{h^2}{3}u'''(x_0) + \ldots$$

SET $\quad u_0 = \frac{4}{3}u_1 - \frac{1}{3}u_2$

$$u''(x_1) \approx \frac{\frac{4}{3}u_1 - \frac{1}{3}u_2 - 2u_1 + u_2}{h^2}$$

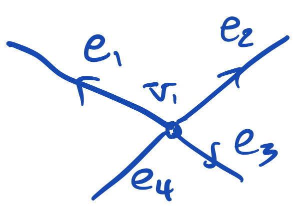
$$= \frac{-\frac{2}{3}u_1 + \frac{2}{3}u_2}{h^2}$$

$$D_2'' = \frac{1}{h^2}\begin{bmatrix} -\frac{2}{3} & \frac{2}{3} & & & \\ 1 & -2 & 1 & & \\ & & \ddots & & \\ & & 1 & -2 & 1 \\ & & & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

$D_2''\vec{u} = \vec{f}$ $\qquad$ 2ND ORDER

BUT D_2'' NOT SYMMETRIC, PREFER TO PRESERVE THE SELF-ADJOINTNESS OF $\frac{d^2}{dx^2}$

LOOKING AHEAD, WANT TO SATISFY VERTEX CONDITIONS AT VERTEX ATTACHED TO SEVERAL EDGES



$$u^{(1)}(0) = u^{(2)}(0) = u^{(3)}(0) = u^{(4)}(0) = v_1$$

$$\sum_{j=1}^{4} w_j\, u^{(j)\prime}(0) + \alpha v_1 = 0$$

OUR PREVIOUS STRATEGY:

- SOLVE FOR v_1 IN TERMS OF NEARBY POINTS
- USE THIS TO ELIMINATE v_1 FROM THE SYSTEM

WILL BE UNWIELDY

GOING FORWARD, WANT TO LEAVE v_1 AS AN UNKNOWN

MODEL PROBLEM

$$\begin{cases} u'' = f & 0 < x < 1 \\ u'(0) + \alpha u(0) = \varphi_0 \\ -u'(1) + \alpha u(1) = \varphi_1 \end{cases}$$

GHOST POINTS: IDEA INTRODUCE ADDITIONAL POINTS OUTSIDE THE DOMAIN

$$x_j = \frac{j - \frac{1}{2}}{N}, \quad j = 0, \dots, N+1$$

2 GRIDS: EXTENDED GRID $\{x_0, \dots, x_{N+1}\}$

INTERIOR GRID $\{x_1, \dots, x_N\}$

FOR A TOTAL OF $N+2$ UNKNOWNS

APPROXIMATE u'' AT $x_1, \dots, x_N$

$$\begin{pmatrix} u''(x_1) \\ \vdots \\ u''(x_N) \end{pmatrix} \approx \frac{1}{h^2} \underbrace{\begin{pmatrix} 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & & \ddots & & & \\ & & & & 1 & -2 & 1 & \\ & & & & & 1 & -2 & 1 \end{pmatrix}}_{N \times (N+2) \text{ RECTANGULAR MATRIX: } L_{INT}} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \\ u_{N+1} \end{pmatrix}$$

L HAS THE CORRECT NULLSPACE: $V_i = ai + b$ THEN $L\vec{v} = 0$

Let $$P_{INT} = \underbrace{\begin{bmatrix} 0 & 1 & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & 1 & 0 \end{bmatrix}}_{N+2} \Big\} N$$ AND $$\vec{f} = \begin{bmatrix} f_0 \\ \vdots \\ f_{N+1} \end{bmatrix}$$

INTERPOLATION TO INTERIOR POINTS

THEN THE ODE PART IS $L_{INT} \vec{u} = P_{INT} \vec{f}$

N EQNS FOR $N+2$ UNKNOWNS

NOW $u(0) = \frac{u_0 + u_1}{2} + O(h^2)$, $u'(0) = \frac{-u_0 + u_1}{h} + O(h^2)$

SO THE LEFT BC IS , UP TO $O(h^2)$,

$$\left(\frac{\alpha_0}{2} - \frac{1}{h}\right) u_0 + \left(\frac{\alpha_0}{2} + \frac{1}{h}\right) u_1 = \varphi_0$$

SIMILARLY AT RIGHT

$$\left(\frac{\alpha_1}{2} + \frac{1}{h}\right) u_N + \left(\frac{\alpha_1}{2} - \frac{1}{h}\right) u_{N+1} = \varphi_1$$

$$M_{BC} = \left.\begin{bmatrix} \frac{\alpha_0}{2} - \frac{1}{h} & \frac{\alpha_0}{2} + \frac{1}{h} & \cdots & 0 & 0 \\ 0 & \underbrace{0 \qquad\qquad}_{N\times 2} & & \frac{\alpha_1}{2} + \frac{1}{h} & \frac{\alpha_1}{2} - \frac{1}{h} \end{bmatrix}\right\} 2$$

$$\boxed{M_{BC}\vec{u} = \begin{bmatrix} \varphi_0 \\ \varphi_1 \end{bmatrix}}$$

2 EQNS FOR N+2 UNKNOWNS

STACK

$$\left(\frac{L_{INT}}{M_{BC}}\right)\vec{u} = \left(\frac{P_{int}}{O_{2\times(N+2)}}\right)\vec{f} + \left(\frac{O_{N\times 2}}{I_{2\times 2}}\right)\vec{\varphi}$$

<u>REMARK:</u> IF WE WERE TO SOLVE FOR u_0 & u_{N+1} IN TERMS OF $u_1, \dots, u_N$ & PLUG BACK IN, MATRIX ON LHS WOULD BE BOTH SYMMETRIC + TRIDIAGONAL

EXTENDING TO QUANTUM GRAPH

SUPPOSE THERE ARE M EDGES $e_1, \dots, e_M$

EDGE m OF LENGTH ℓ_m

DISCRETIZE USING N_m SUBDIVISIONS $h_m = \frac{\ell_m}{N_m}$

EDGE m CONTRIBUTES A BLOCK OF SIZE $N_m \times (N_m + 2)$

THERE REMAIN 2M VERTEX CONDITIONS

IF VERTEX V_m HAS DEGREE d_m

LET $\mathbb{E}_G^m = \{e_{i_1}, \dots, e_{i_{d_m}}\}$

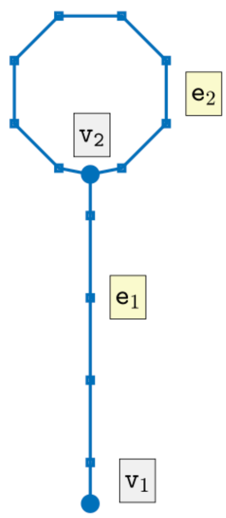
THEN $d_m - 1$ CONTINUITY CONDITIONS

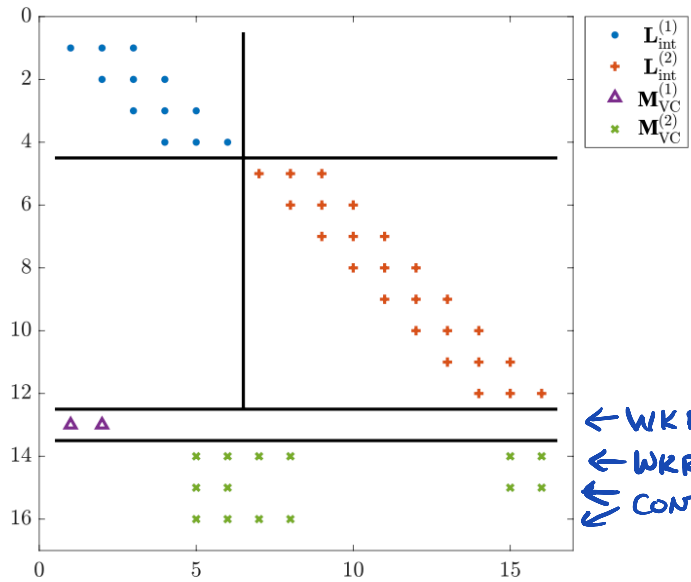
$$e_{i_1}(v_m) = e_{i_2}(v_m) = \dots = e_{i_m}(v_m)$$

+ ONE FLUX BOUNDARY CONDITION

$$\sum_{e_j \in \mathbb{E}_G^m} w_j u_j'(v_m) + \alpha U(v_m) = 0$$

WEIGHTED KIRCHOFF-ROBIN VERTEX CONDS





← WKR AT V_1

← WKR AT V_2

← CONTINUITY AT V_2