LECTURE 2: RECTANGULAR CHEBYSHEV MATRICES

HOW TO GET HIGHER-ORDER SPATIAL DISCRETIZATION?

SIMPLEST (BAD) IDEA: REPLACE 2ND ORDER CENTERED DIFFERENCE OPERATOR WITH HIGHER ORDER

PROBLEM: WIDER STENCIL MAKES IMPLEMENTING BOUNDARY CONDITIONS EVEN HARDER

BETTER: SPECTRAL COLLOCATION

LET $-1 \le x_0 \le \dots \le x_N \le 1$ BE COLLOCATION POINTS

$$u_j = u(x_j) \qquad j = 1, \dots, N$$

$I_N \vec{u}$ = DEGREE-N INTERPOLATING POLYNOMIAL THROUGH (x_j, u_j) $\quad j = 0, \dots, N$

LET $\vec{u} = (u_0, \dots, u_N)^T$

DEFINE A MATRIX $D^{(m)}$ BY $\quad (D^{(m)} \vec{u})_j = u^{(m)}(x)\Big|_{x = x_j}$

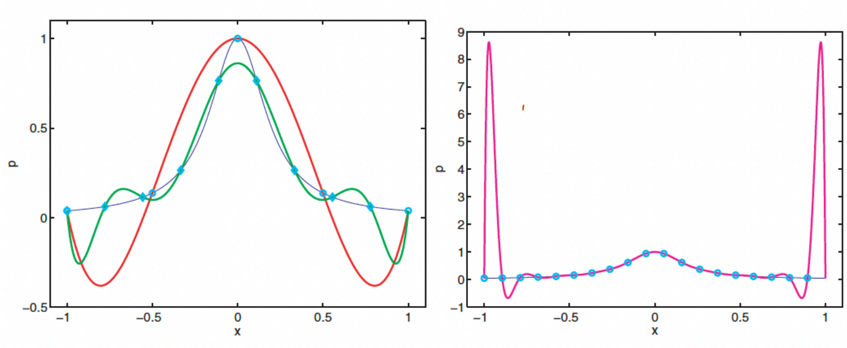
PROBLEM: OBVIOUS CHOICE $x_j = -1 + \frac{2j}{N}$ LEADS TO RUNGE PHENOMENON

$u(x)$

FAMOUS EXAMPLE:

$$u = \frac{1}{1 + 25x^2}$$

DATA & INTERPOLATING POLYNOMIALS I_4, I_9, I_{19}



(a) $n = 4, 9$.

(b) $n = 19$.

REASON $\|u(x) - I_N u(x)\|_\infty \le \left\| \prod_{i=0}^{N} (x - x_i) \right\|_\infty \dfrac{\| u^{(N+1)} \|_\infty}{(N+1)!}$

IF u ANALYTIC
THE FIRST FACTOR $\pi_N(x) = \prod_{i=0}^{N} (x - x_i)$ IS THE PROBLEM AND GROWS FAST FOR LARGE N AND UNIFORMLY-SPACED POINTS

THE FIX: CHOOSE $x_0, \dots, x_N$ TO MINIMIZE FIRST FACTOR

THIS IS ACCOMPLISHED USING THE CHEBYSHEV POLYNOMIALS)
(TCHEBYCHEV Ā FRANCE)

RECALL THE CHEBYSHEV POLYNOMIAL ARE

$T_n(x) = \cos(n \cdot \cos^{-1} x)$

$T_0 = 1$

$T_1 = x$

$$\boxed{\begin{aligned} \cos((n+1)t) &= \cos nt \cos t - \sin nt \sin t \\ \cos((n-1)t) &= \cos nt \cos t + \sin nt \sin t \\ \hline \cos(n+1)t + \cos(n-1)t &= 2 \cos nt \cos t \\ t &= \cos^{-1} x \end{aligned}}$$

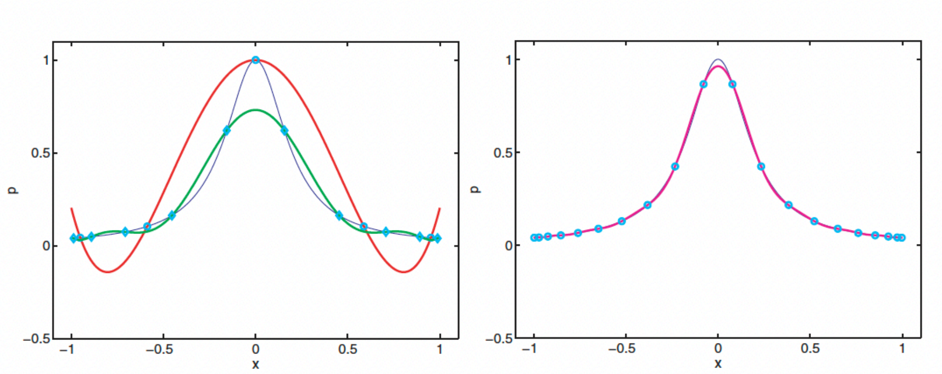
$T_{n+1} = 2x T_n(x) - T_{n-1}(x) \implies T_n$ IS A DEGREE n POLYNOMIAL WITH LEADING COEFFICIENT 2^{n-1}

T_n HAS n ROOTS AT THE CHEBYSHEV POINTS OF THE 1ST KIND $x_k = \cos \dfrac{(2k-1)\pi}{2N}$, $k = 1, \dots, N$

T_n HAS $n+1$ EXTREMA AT 2ND KIND CHEBYSHEV POINTS
$\tilde{x}_k = \cos \dfrac{k\pi}{N}$, $k = 0, \dots, N$, $T_n(\tilde{x}_k) = \pm 1$

USING 1ST-KIND CHEBYSHEV POINTS
$|\pi_N| < \dfrac{1}{2^N}$

THEN APPROXIMATION IS SPECTRAL $= o(N^{-p}) \quad \forall p$



(a) $n = 4, 9$.   (b) $n = 19$.

WE WANT TO MIMIC OUR APROACH FROM THE CENTERED DIFFERENCES. WE WANT AN $N \times (N+2)$ MATRIX THAT MAPS LINEARLY-VARYING VECTORS TO ZERO

FOLLOW A PAPER BY DRISCOLL + HALE

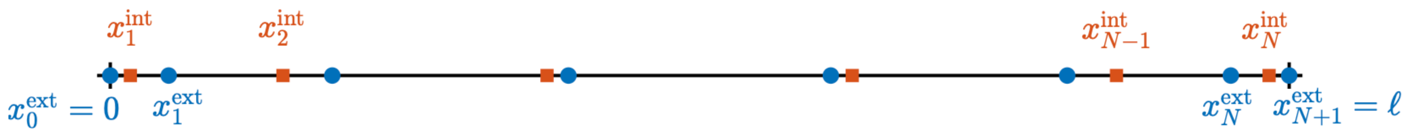
TWO GRIDS

$$x_k^{ext} = \frac{\ell}{2} \cos \frac{k\pi}{N+1} \qquad k = 0, \dots, N+1$$

2ND-KIND CHEBYSHEV POINTS

$$x_k^{int} = \frac{\ell}{2} \cos \frac{(2k-1)\pi}{N} \qquad k = 1, \dots N$$

1ST KIND CHEBYSHEV POINTS



STRATEGY: DEFINE THE $(N+2)\times(N+2)$ COLLOCATION 2ND DERIVATIVE MATRIX $D^{(2)}$, DEFINED ON $\vec{x}^{\,ext}$

RESAMPLE THIS TO $\vec{x}^{\,int}$ BY EVALUATING THE CHEBYSHEV INTERPOLANT AT THESE POINTS

RECALL, GIVEN POINTS $x_0, \ldots, x_N$

DEFINE $$L_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{N} \frac{x - x_j}{x_i - x_j}$$ THE LAGRANGE POLYNOMIALS OF DEGREE N

Then $L_j(x_i) = \delta_{ij}$

SO GIVEN DATA $y_0, \ldots, y_N$

$$P_N(x) = \sum_{j=0}^{N} y_j L_j(x)$$

INTERPOLATES THE POINTS (x_j, y_j) $j=0,\ldots,N$

Let $$\psi(x) = \prod_{i=0}^{N} (x - x_j) \quad \Rightarrow \quad L_j(x) = \frac{w_j \psi(x)}{x - x_j}$$

$$w_j = \frac{1}{\prod_{\substack{i=0 \\ i \neq j}}^{N} (x_i - x_j)} \qquad P_N(x) = \sum \frac{w_j y_j \psi(x)}{x - x_j}.$$

$$= \psi(x) \sum \frac{w_j y_j}{(x - x_j)}$$

Letting $y_j = 1$, $j = 0, \ldots, N$

WE GET $$1 = \psi(x) \sum \frac{w_j}{(x - x_j)} \Rightarrow \psi = \frac{1}{\sum \frac{w_j}{x - x_j}}$$

So INTERPOLATE FROM x^{ext} to x^{int} USING

$$w_k = \prod_{\substack{j=0 \\ k \neq j}}^{N+1} \left(x_k^{ext} - x_j^{ext} \right) \qquad k = 0, \dots, N+1$$

$$p_{N+1}(x) = \frac{\displaystyle\sum_{k=0}^{N+1} \frac{w_k}{x - x_k^{ext}} \, y_k}{\displaystyle\sum_{k=0}^{N+1} \frac{w_k}{x - x_k^{ext}}}$$

EVALUATING THIS AT $\vec{x}^{INT} = [x_1^{INT} \dots x_N^{INT}]^T$

WE FIND $p_{N+1}(\vec{x}^{INT}) = P_{INT} \cdot p_{N+1}(\vec{x}^{EXT})$

$$(P_{int})_{jk} = \frac{w_k}{x_j^{int} - x_k^{ext}} \left(\sum_{\ell=0}^{N+1} \frac{w_\ell}{x_j^{int} - x_\ell^{ext}} \right)^{-1} \Big\} N \times (N+2)$$

DEFINE OUR 2ND DERIVATIVE MATRIX BY

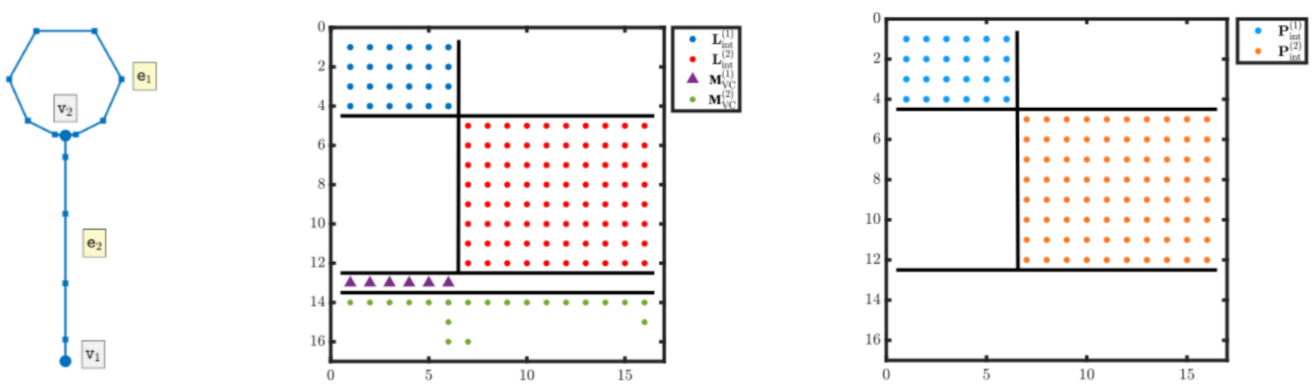
$$L_{INT} = P_{INT} D^{(2)}$$

+ 2 BOUNDARY CONDITIONS

TO SOLVE $u'' = f$

DISCRETIZED GRAPH

$$L_{VC} = \begin{bmatrix} L_{INT} \\ M_{VC} \end{bmatrix} \qquad P_0 = \begin{bmatrix} P_{INT} \\ 0_{2|E_G| \times (N+2)} \end{bmatrix}$$



(a) (b) (c)

BIGGEST TAKEAWAY: $L\vec{u} = P\vec{f}$, $M_{BC}u = \vec{\Phi}$

ON EACH EDGE k DEFINE 2 GRIDS

$\vec{x}^{ext}$ OF N_k+2 POINTS

$\vec{x}^{int}$ OF N_k POINTS

(1) DATA GIVEN ON $\vec{x}^{ext}$ BUT ALL EDGE-DEFINED EQNS EVALUATED ON $\vec{x}^{INT}$.

(2) SIMULTANEOUSLY, THE UNKNOWN IN ANY EQUATION WE SET UP MUST ALSO SOLVE THE DISCRETIZED VERTEX CONDITIONS.

(SEMI) INFINITE EDGES (I DIDN'T GET TO THIS IN LECTURES, BUT IT'S INTERESTING/USEFUL)

CONSIDER NLS ON A HALF LINE

$$\Lambda\psi + \psi'' + 2\psi^3 = 0 \qquad 0 < x < \infty$$

$$\psi'(0) + \alpha\psi(0) = 0$$

$$\lim_{x\to\infty} \psi(x) = 0 \qquad \text{(AT WHAT RATE?)}$$

HOW DO WE COMPUTE THIS ACCURATELY?

FIRST ATTEMPT: TRUNCATE $[0,\infty)$ TO $[0,L]$

APPLY DIRICHLET B.C AT $x=L$

WHAT KIND OF DISCRETIZATION?

START WITH UNIFORM, 2ND ORDER CENTERED DIFF.

let $h = \frac{L}{N}$, $x_k = kh$, $k=0,..,N$ $\qquad \psi_k \approx \psi(x_k)$

$$\psi_N = 0$$

NOW THERE ARE TWO SOURCES OF ERROR WHICH WE WANT TO BALANCE

① DISCRETIZATION ERROR $\mathcal{E}_{DISC} \propto h^2 \max_{0 \le x \le L} |\psi''(x)| = \frac{L^2}{N^2} \|\psi''\|_\infty$

② TRUNCATION ERROR $\mathcal{E}_{TRUNC} \propto \psi_{EXACT}(L)$

- IF $\psi(x)$ DECAYS SLOWLY, THEN MUST TAKE L LARGE, THEN MUST TAKE N LARGE ENOUGH TO MAKE $\mathcal{E}_{DISC}$ COMPARABLE
- LARGE $N \Rightarrow$ LARGE MATRICES, SLOW COMPUTATION
- WASTEFUL BECAUSE THE VALUES OF ψ_k NEAR $x=L$ CONTRIBUTE LITTLE TO APPROXIMATION OR ERROR

ONE SOLUTION

Let $s \in [0,1]$, DISCRETIZE $h = \frac{1}{N}$, $s_k = kh$

$$x = f(s)$$
$$x_k = f(s_k)$$
$$L = f(1)$$

$f(0) = 0$, $f'(0) > 0$ ON $[0,1]$

COMMON CHOICE $f(s) = \sinh Ms$
$L = \sinh M$

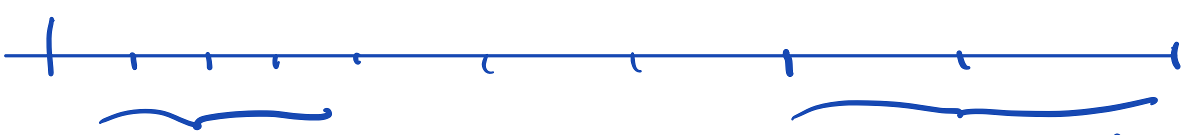
$$\frac{d}{dx} = \frac{\frac{d}{ds}}{\frac{dx}{ds}} = \frac{1}{f'(s)} \frac{d}{ds} = g(s) \frac{d}{ds} \Rightarrow \frac{d}{dx} = \frac{1}{M \cosh Ms} \frac{d}{ds}$$

FIND

$$\frac{d^2}{dx^2} = g \cdot g' \frac{d}{ds} + g^2 \frac{d^2}{ds^2}$$

$$\frac{d^2}{dx^2} = \frac{\operatorname{sech}^2 Ms}{M^2} \frac{d^2}{ds^2} - \frac{1}{M} \operatorname{sech}^2 Ms \tanh Ms \frac{d}{ds}$$

SOME THEORY TO SHOW HOW TO CHOOSE M



FOR CHEBYSHEV, THIS ISN'T ENOUGH TO COUNTERACT CLUSTERING NEAR RIGHT ENDPOINT

$$x = \sinh M\left(\left(1 - \sqrt{1-s}\right)\right)$$

KEEPS CLUSTERING NEAR $x = s = 0$

SPREADS OUT POINTS NEAR $s = 1$, $x = \sinh M$