LECTURE 5: CONTINUATION METHODS

THIS LECTURE BASED MAINLY ON NAYFEH-BALACHANDRAN CH 6

OUR MAIN INTEREST IS SOLUTIONS TO

$$-i\psi_t + \Delta\psi + 2|\psi|^2\psi = 0 \quad \text{ON A QG } \mathcal{G}$$

$$\psi = e^{i\Lambda t}\Phi$$

$$\Lambda\Phi + \Delta\Phi + 2|\Phi|^2\Phi$$

CONTINUATION FUNDAMENTALLY DEPENDS ON THE IMPLICIT FUNCTION THEOREM

Let $F \in C^r(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$, $r \geq 1$ AS NEEDED

$F = F(x, \mu)$, $x \in \mathbb{R}^n$ STATEVECTOR
$\mu \in \mathbb{R}^m$ PARAMETER VECTOR

LET x_0 SOLVE (*) $F(x,\mu) = 0$ FOR $\mu = \mu_0$

ASSUME THE JACOBIAN $D_x F(x_0, \mu_0)$ NONSINGULAR

THEN $\exists$ BALL $B_\delta(x_0, \mu_0) \subset \mathbb{R}^{n+m}$ AND A C^r FUNCTION $G(\mu): \mathbb{R}^m \to \mathbb{R}^n$ WITH $G(\mu_0) = x_0$ s.t. $F(G(\mu), \mu)$ IS THE UNIQUE SOLUTION TO (*) IN THE BALL

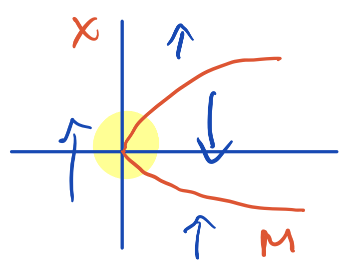
WE'LL ASSUME $m=1$, SO SCALAR PARAMETER

POINTS (x_0, μ_0) WHERE $D_x F(x_0, \mu_0)$ IS SINGULAR ARE CALLED BIFURCATION POINTS

THREE BASIC EXAMPLES

(1) $\dfrac{dx}{dt} = x^2 - \mu$ SO $F(x;\mu) = x^2 - \mu$

SOLUTIONS $x = \pm\sqrt{\mu}$ $\quad \mu \geq 0$



LOOK ON BRANCH $x = \sqrt{\mu}$

$$\frac{\partial F}{\partial x} = 2x$$

$$\frac{\partial F}{\partial x}(\sqrt{\mu}, \mu) = 2\sqrt{\mu} \neq 0 \quad \text{UNLESS } x = \mu = 0$$

NEAR $(0,0)$ BOTH EXISTENCE & UNIQUENESS FAIL

CALLED THE SADDLE-NODE BIFURCATION, FOLD, OR TURNING POINT

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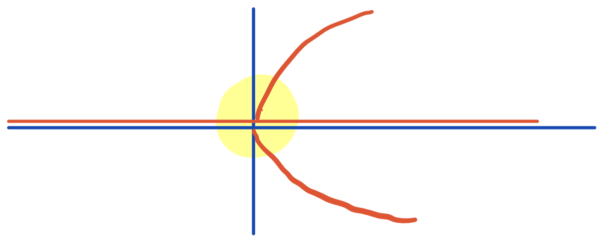
(2) $\frac{dx}{dt} = \mu x - x^3$ SOLUTIONS $x=0$, $x = \pm\sqrt{\mu}$

$F = \mu x - x^3$

$F_x = \mu - 3x^2$

$F_x(0, \mu) = \mu$

$F_x(\pm\sqrt{\mu}, \mu) = -2\mu$



BIFURCATION AT $(0,0)$, UNIQUENESS FAILS

CALLED A PITCHFORK BIFURCATION OR SYMMETRY-BREAKING BIFURCATION

NOTE $f(-x, \mu) = -f(x, \mu)$

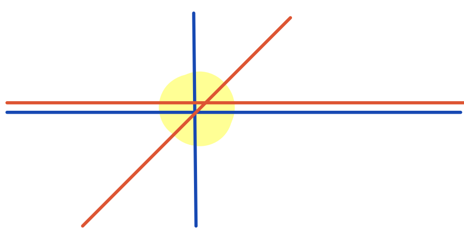
THE SOLUTION $x = 0$ IS INVARIANT TO THE SYMMETRY $x \to -x$

THE SOLUTIONS $x = \pm\sqrt{\mu}$ ARE NOT BUT THE FORM A GROUP ORBIT OF THIS SYMMETRY

(3) $\frac{dx}{dt} = F(x, \mu) = x^2 - \mu x$ $x = 0$, $x = \mu$

$F_x = 2x - \mu$

$$\left.\begin{array}{l} F_x(0,\mu) = -\mu \\ F_x(\mu,\mu) = \mu \end{array}\right\} \text{BOTH VANISH AT } (0,0)$$



THE TRANSCRITICAL BIFURCATION. EXAMPLES (2) & (3)

DEMONSTRATE BRANCH POINTS

THE ABOVE EXAMPLES ARE GENERIC UNDER CERTAIN ASSUMPTIONS ON $F(x_0, \mu_0)$ & $D_x F(x_0, \mu_0)$ + HIGHER ORDER TERMS

THEY ARE ESSENTIALLY SAME IN HIGHER DIMENSIONS + CAN BE JUSTIFIED RIGOROUSLY BY EXPANDING $F(x, \mu)$ IN A NBHD OF A BIFURCATION POINT

ANY PROGRAM COMPUTING BRANCHES OF SOLUTIONS MUST BE ABLE TO HANDLE FOLDS + BRANCHES

Let's CALL OUR PARAMETER α

$$F(x; \alpha) = 0$$

FOR EACH α, THERE MAY BE MULTIPLE SOLUTIONS BUT IF AT (x_0, α_0), $D_x F$ NONSINGULAR, UNIQUE SOLUTION NEARBY

PARAMETER CONTINUATION

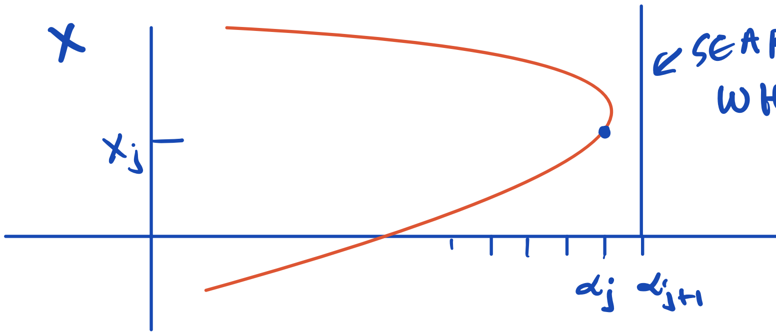
GIVEN SOL'N x_0, α_0

Let $\alpha_j = \alpha_0 + j\Delta\alpha$

IF SOLUTIONS FOUND FOR $\alpha_0, \dots, \alpha_j$

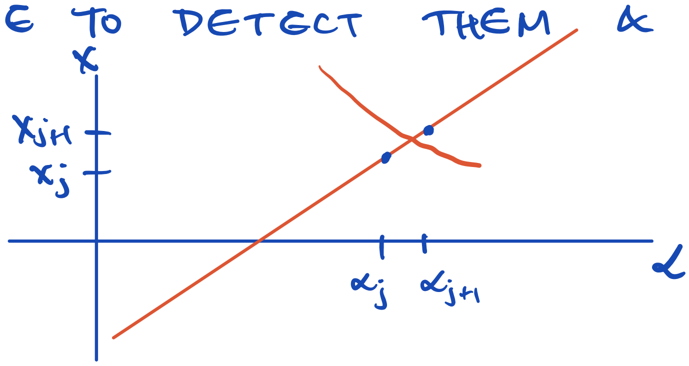
FOR $\alpha = \alpha_{j+1}$, USE NEWTON'S METHOD TO SOLVE FOR x_{j+1} USING x_j AS AN INITIAL CONDITION. SINCE $|\alpha_j - \alpha_{j+1}| << 1$, MUST HAVE $\|x_{j+1} - x_j\| << 1$, SHOULD CONVERGE QUICKLY

PROBLEM: THIS FAILS AT TURNING POINTS



← SEARCHING FOR X WHERE NO NEARBY X EXISTS

CAN USUALLY JUMP OVER BRANCH POINTS, BUT WOULD LIKE TO DETECT THEM & SWITCH BRANCHES



PARTIAL FIX: ARCLENGTH CONTINUATION

LET $x = x(s)$, $\alpha = \alpha(s)$, ASSUME THIS IS AN ARC-LENGTH PARAMETERIZATION

$$F(x(s), \alpha(s)) = 0 \qquad (1)$$

DIFFERENTIATE WRT s

$$F_x(x,\alpha)\, x' + F_\alpha(x,\alpha)\,\alpha' = 0 \qquad (2)$$

CAN WRITE AS

$$n\left\{ \underbrace{\left[F_x \mid F_\alpha \right]}_{n+1} \begin{bmatrix} x' \\ \alpha' \end{bmatrix} \right. = 0$$

ONE MORE CONDITION NEEDED FOR UNIQUENESS

$$\|x'\|^2 + (\alpha')^2 = 1 = x'^T x + \alpha'^2 \qquad (3)$$

TO SOLVE (2) FIRST SOLVE

$$F_x(x,\alpha)\, z = -F_\alpha(x,\alpha) \qquad (4)$$

THEN BY LINEARITY $x' = z \cdot \alpha'$ (5)

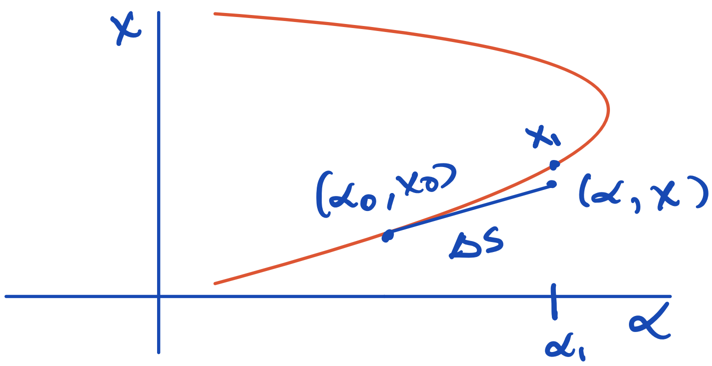
SUBSTITUTE (5) INTO (3)

$$(6) \qquad \alpha' = \pm (1 + z^T z)^{1/2}$$ (CAN CONTINUE IN EITHER DIRECTION)

THEN CAN CONTINUE x AND α USING FWD EULER

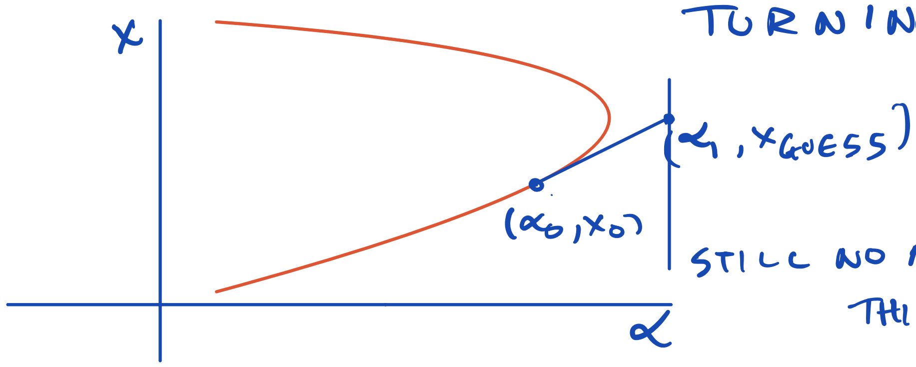
$$x = x_0 + x' \Delta s$$
$$\alpha = \alpha_0 + \alpha' \Delta s$$

THIS CONSTITUTES A NEW PREDICTION ALONG THE TANGENT



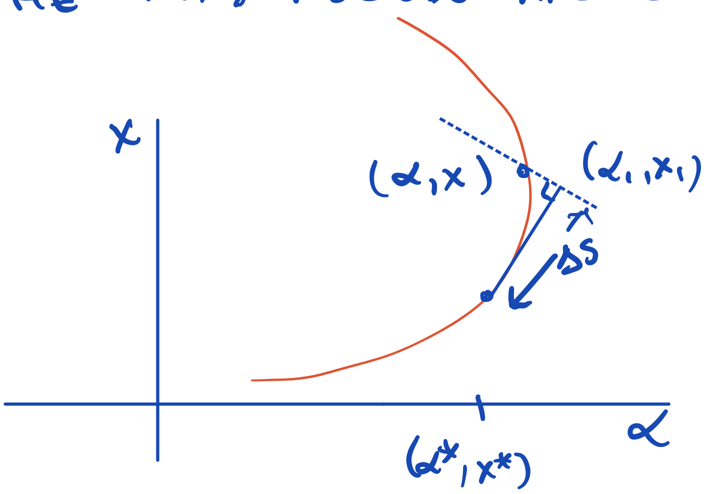
TANGENT PREDICTOR

THIS ALONE IS INSUFFICIENT TO GO AROUND TURNING POINTS



STILL NO NEARBY SOLUTION AT THIS VALUE OF α

THE FIX: PSEUDO-ARCLENGTH CONTINUATION



MAY NEED TO ADAPT Δs FOR A SOLUTION TO EXIST

TO COMPUTE NEW POINT

PREDICT
$$\alpha_1 = \alpha^* + \alpha^{*\prime} \Delta s$$
$$x_1 = x^* + x^{*\prime} \Delta s$$

$$\begin{pmatrix} x - x_1 \\ \alpha - \alpha_1 \end{pmatrix}^T \begin{pmatrix} x_1 - x^* \\ \alpha_1 - \alpha^* \end{pmatrix} = 0 \qquad (7)$$

AND $\quad F(x,\alpha) = 0$

(7) CAN BE WRITTEN

$$(x - x^*)^T x^{*\prime} + (\alpha - \alpha^*)\alpha^{*\prime} - \underbrace{(\alpha^{*\prime 2} + (x^{*\prime})^T x^{*\prime})}_{=1}\Delta s = 0$$

$$g(x,\alpha) = (x - x^*)^T x^{*\prime} + (\alpha - \alpha^*)\alpha^{*\prime} - \Delta s = 0 \qquad (8)$$

APPLY NEWTON'S METHOD TO (1) & (8)

$$x^{k+1} = x^k + \Delta x^{k+1}$$
$$\alpha^{k+1} = \alpha^k + \Delta\alpha^{k+1}$$

NOTE SUPERSCRIPTS REFER TO ITERATION #

$$F_x(x^k,\alpha^k)\Delta x^{k+1} + F_\alpha(x^k,\alpha^k)\Delta\alpha^{k+1} = -F(x^k,\alpha^k) \qquad (9)$$
$$(x^{*\prime})^T \Delta x^{k+1} + \alpha^{*\prime}\Delta\alpha^{k+1} = -g(x^k,\alpha^k) \qquad (10)$$

TO SOLVE (9), SOLVE TWO SYSTEMS

$$F_x(x^k,\alpha^k) z_2 = -F_\alpha(x^k,\alpha^k)$$

$$F_x(x^k,\alpha^k) z_1 = -F(x^k,\alpha^k)$$

THEN BY LINEARITY $\quad \Delta x^{k+1} = z_1 + z_2 \Delta\alpha^{k+1} \qquad (11)$

SUBSTITUTE INTO (10), YIELDING

$$\Delta\alpha^{k+1} = -\frac{(g(x^k,\alpha^k) + z_1^T x^{*\prime})}{\alpha^{*\prime} + z_2^T x^{*\prime}} \qquad (12)$$

THEN SUBSTITUTE (12) INTO (11) TO GET Δx^{k+1}

"BORDERED NEWTON METHOD"

WHAT ELSE WOULD WE WANT SUCH A METHOD TO DO?

- ADAPT: TAKE LARGE STEPS WHEN BRANCH CURVATURE SMALL & SMALL STEPS WHEN LARGE
- DETECT FOLD AND BRANCH POINTS
- SWITCH BRANCHES AT BRANCH POINTS
- ORGANIZE THE DATA IN A USEFUL FORM

NOTE TO SELF: RUN THE CONTINUATION INSTRUCTIONS FILE BEFORE LECTURE SO I CAN POP OUT IMAGES

SOMETHING HARD: ALGORITHMS FOR DETECTING BRANCHES + COMPUTING BRANCHING DIRECTIONS WHEN $D_x F$ HAS NULL SPACE WITH DIMENSION >1

OF BIG INTEREST FOR QUANTUM GRAPHS WHERE BIG SYMMETRY GROUPS $\Longrightarrow$ BIG NULL SPACE

$\Longrightarrow$ NON-GENERIC BIFURCATIONS

NOW THAT I'VE GIVEN YOU A SENSE OF WHAT QGLAB CAN DO, WHAT ELSE WOULD BE MOST USEFUL?